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ASYMPTOTIC FORMULA FOR THE NATURAL FREQUENCIES OF NONCIRCULAR CYLINDRICAL FLUID-FILLED SHELLS*

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An exact asymptotic formula is proved for the natural frequencies of noncircular cylindrical shells. It turns out that the spectrum dissociates asymptotically into four series corresponding to different kinds of state of stress. Natural frequency modes are written down in the form of rapidly oscillating functions corresponding to a quasi-transverse state of stress.

1. A cylindrical shell of arbitrary outline with elastic flat bulkheads is considered, whose plane is orthogonal to cylinder generatrix. The vessel thus obtained is filled entirely with fluid.

We assume that the velocity potential Γ_1 on the bulkheads is zero, and simple support conditions are satisfied for the displacements u, v, w on the cylindrical shell boundary Γ (α is the generatrix arc length)

$$\varphi|_{\Gamma_1}=0,\quad \left.\frac{\partial u}{\partial\alpha}\right|_{\Gamma}=0,\quad v|_{\Gamma}=0,\quad w|_{\Gamma}=0,\quad \left.\frac{\partial^2 w}{\partial\alpha^2}\right|_{\Gamma}=0 \tag{1.1}$$

Then the problem of determining the natural frequencies of the combined oscillations of the mechanical system "cylinder side surface—fluid" results in the following system of equations

$$\begin{gathered}
-\left[\frac{\partial^2}{\partial\alpha^2}+\frac{1-\sigma}{2}\frac{\partial^2}{\partial\beta^2}\right]u-\frac{1+\sigma}{2}\frac{\partial^2 v}{\partial\alpha\partial\beta}+\frac{\sigma}{R(\beta)}\frac{\partial w}{\partial\alpha}=\lambda u \\
-\frac{1+\sigma}{2}\frac{\partial^2 u}{\partial\alpha\partial\beta}-\left[\frac{\partial^2}{\partial\beta^2}+\frac{1-\sigma}{2}\frac{\partial^2}{\partial\alpha^2}\right]v+\frac{\partial}{\partial\beta}\left(\frac{w}{R(\beta)}\right)=\lambda v \\
-\frac{\sigma}{R(\beta)}\frac{\partial u}{\partial\alpha}-\frac{1}{R(\beta)}\frac{\partial v}{\partial\beta}+\left[\frac{1}{R^2(\beta)}+\frac{h^2}{12}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)^2\right]w=\lambda w-\frac{i\omega\rho_f}{Eh}\varphi|_{\Gamma_2} \\
(\partial\varphi/\partial n)_{\Gamma_2}=i\omega w,\quad \Delta\varphi+\omega^2c^{-1}\varphi=0\quad (\lambda=(1-\sigma^2)\omega^2\rho_s/E)
\end{gathered} \tag{1.2}$$

The first four equations are satisfied on the cylinder side surface, and the last, the Helmholtz equation, within the vessel, where u, v, w, φ satisfy the conditions (1.1). The system (1.2) is written in conformity with the notation used in /1/; we just note that ρ_f is the fluid density, ρ_s the shell density, $\partial/\partial n$ is differentiation in the external normal direction, and Γ_2 is the cylinder side surface.

Theorem 1. The spectrum of the problem (1.1), (1.2) is real, discrete, and has a unique limit at infinity. The eigennumbers ω are symmetric relative to zero.

Proof. The substitution

$$u=u_1,\ v=v_1,\ w=\rho_1\sqrt{h}w_1,\ \varphi=\rho_1\sqrt{h}\varphi\ (\rho_1=\sqrt{E/\rho_f})$$

sets the system (1.2) in a quadratic bundle relative to the spectral parameter ω ($B\geqslant 0$ and C are the Hermitian matrices)

$$A\mathbf{x}=\omega^2B\mathbf{x}+\omega C\mathbf{x},\quad \mathbf{x}=(u_1, v_1, w_1, \varphi_1|_{\Gamma}, \varphi_1|_{\Omega}) \tag{1.3}$$

Following /2/, it is natural to introduce the space L of vector functions

$$\mathbf{x}: L=L_2(\Gamma)\oplus L_2(\Gamma)\oplus L_2(\Gamma)\oplus L_2(\Gamma)\oplus L_2(\Omega)$$

$$(\mathbf{x}_1,\mathbf{x}_2)=\int_{\Gamma}(u_1^1u_1^2+v_1^1v_1^2+w_1^1w_1^2+\varphi_1^1|_{\Gamma}\varphi_1^2|_{\Gamma})\,d\Gamma+\int_{\Omega}\varphi_1^1\varphi_1^2\,d\Omega$$

It can be shown that the operator A in (1.3) is symmetric and nonnegative in the subspace of smooth vector functions. As is shown in /2/, the operator A has a self-adjoint closure in the space L with a completely continuous resolvent. The discreteness of the spectrum of the self-adjoint bundle (1.3) follows from Theorem 1.5.1 in /3/.

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If x_0 is an eigenvector corresponding to the eigenvalue ω_0, then

$$\omega_0^2 (Bx_0, x_0) + \omega_0 (Cx_0, x_0) - (Ax_0, x_0) = 0 \tag{1.4}$$

holds.

By virtue of self-adjointness, the bundles of coefficients of (1.4) are real. The value of ω_0 is real if $(Bx_0, x_0) = 0$. Let $(Bx_0, x_0) \neq 0$, then that ω_0 is real follows from the fact the $(Ax_0, x_0) \geqslant 0, (Bx_0, x_0) \geqslant 0$, i.e., the discriminant of (1.4) is positive. The last assertion of the theorem is verified directly.

The system (1.3) depends regularly on the small parameter h. For $h = 0$ it separates into two unrelated problems. The first, corresponding to the first two equations in (1.2), is, in combination with the second and third boundary conditions (1.1) an eigenvalue problem for plane oscillations of the cylindrical shell and has a nonnegative discrete spectrum with a single limit point at infinity. The second problem has the form

$$-i\omega\rho_f E^{-1} \varphi |_{\Gamma_2} = 0, \ (\partial\varphi / \partial n)_{\Gamma_2} = i\omega w, \ \Delta\varphi + \omega^2 c^{-1} \varphi = 0 \tag{1.5}$$
$$\varphi |_{\Gamma_1} = 0$$

At the point $\omega = 0$ the system (1.5) has an infinite proper subspace of vector functions of the form $(w, 0, 0)$, where, $w \in L_2(\Gamma_2)$ is an arbitrary function, and also the eigenfunction $(0, 1, 1)$. For $\omega \neq 0$, the problem (1.5) is equivalent to the Dirichlet problem for the Helmholtz equation. As will be shown below, the spectrum of the problem (1.1), (1.2) is closely related to the spectrum of the problem mentioned and the problem (1.5), hence, the role of the point $\omega = 0$ is analogous to the role of a continuous spectrum in dry shells: it is the limit point for the eigennumbers of the problem (1.1), (1.2) as $h \to 0$. We shall later call the eigenvalue problem (1.1), (1.2) a moment problem.

Remarks. 1°. Zero is an eigenvalue of the moment problem, hence the eigenvectors equal $(0, 0, 0, 1, 1)$ and $(1, 0, 0, 0, 0)$.

2°. All the above is valid for an arbitrary closed shell filled completely with fluid.

2. The substitution

$$u = u(s) \cos \varkappa\alpha, \quad v = v(s) \sin \varkappa\alpha, \quad w = w(s) \sin \varkappa\alpha, \quad \varphi = \varphi(x, y) \sin \varkappa\alpha, \quad \varkappa = k\pi / l$$

(l is the cylinder length, and x, y are orthogonal coordinates in the transverse section) reduces the system (1.2) to the following system.

$$-\left[\frac{1-\sigma}{2}\frac{d^2}{d\beta^2} - \varkappa^2\right] u - \frac{1+\sigma}{2}\varkappa\frac{dv}{d\beta} + \frac{\sigma}{R(\beta)}\varkappa w = \lambda u \tag{2.1}$$
$$\frac{1+\sigma}{2}\varkappa u - \left[\frac{d^2}{d\beta^2} - \frac{1-\sigma}{2}\varkappa^2\right] v + \frac{d}{d\beta}\left(\frac{w}{R(\beta)}\right) = \lambda v$$
$$\frac{\sigma}{R(\beta)}\varkappa u - \frac{1}{R(\beta)}\frac{dv}{d\beta} + \left[\frac{1}{R^2(\beta)} + \frac{h^2}{12}\left(\frac{d^2}{d\beta^2} - \varkappa^2\right)^2\right] w = \lambda w - \frac{i\omega\rho_f}{Eh}\varphi|_\Gamma$$
$$\Delta\varphi + (\omega^2 c^{-1} - \varkappa^2)\varphi = 0, \ (\partial\varphi / \partial n)_\Gamma = i\omega w$$

We shall henceforth consider $R(\beta)$ an infinitely differentiable function.

Let us describe the method which can be used to reduce the problem (2.1) to a system of pseudodifferential equations on the contour Γ. Let $T(\lambda)$ denote a linear operator that sets the derivative in the direction of the external normal in correspondence with the boundary value of the solution of the Helmholtz equation. The operator $T(\lambda)$ can be represented in the following form (a is the length of the cylinder directrix):

$$T(\lambda) = D + T_1(\lambda), \quad D\left(\exp\frac{2n\pi i\beta}{a}\right) = \frac{2|n|\pi}{a}\exp\frac{2n\pi i\beta}{a} \tag{2.2}$$

Here $T_1(\lambda)$ is an integral operator of order -1, whose kernel is a meromorphic function of λ with poles at points of the spectrum of the problem

$$\Delta\varphi + (\omega^2 c^{-1} - \varkappa^2)\varphi = 0, \ \varphi|_\Gamma = 0 \tag{2.3}$$

A representation of the operator $T(\lambda)$ in the form (2.2) can be obtained by calculating its symbol (see /4/). This can be done, for instance, by using the explicit form of the fundamental solution of the Helmholtz equation. Hence, the principal symbol of the operator $T(\lambda)$ turns out to equal $|\xi|$. If D denotes the operator with such a symbol, then it can easily be shown that the second relation in (2.2) holds for D. The meromorphicity of $T_1(\lambda)$ as a function of λ follows from the definition of the operator $T(\lambda)$. Let us note the operator

$T_1(\lambda)$ is representable in the form

$$T_1(\lambda)\,w = T_2(\lambda)\,w + \sum_{\lambda_i \in G} \frac{f_i(\beta)\,(f_i,\,w)}{\lambda_i - \lambda} \tag{2.4}$$

where λ_i is a pole of the kernel $T_1(\lambda)$ in the domain G of the complex plane. The operator $T_2(\lambda)$ is analytic in G, and $f_i(\beta)$ are smooth functions on Γ. We express u, v in terms of w from the first two equations in (2.1), and we substitute them into the third equation. We express w in terms of $\varphi|_\Gamma$ from the last two equations in (2.1) by using the operator $T(\lambda): w = (i\omega)^{-1}\, T(\lambda)\,\varphi$.

Taking account of (2.4), we then conclude that the system (2.1) is equivalent to the following equation:

$$\frac{h^3}{12}\left[\frac{d^2}{d\beta^2} - \varkappa^2\right]^2 (D + T_2(\lambda))\,\varphi + hK(\lambda)\,\varphi = \lambda\,[h\,(D + T_2 + \rho E)]\,\varphi, \quad \rho = \frac{\rho_f}{(1-\sigma^2)\,\rho_s} \tag{2.5}$$

Here $K(\lambda)$ is an integral operator with kernel meromorphic in λ, where $K(\lambda)$ can be selected in conformity with (2.4) so that the meromorphic part of $K(\lambda)$ would be finite on the segment $I = [\lambda_1, \lambda_2]$, and the operator $T_2(\lambda)$ is regular.

The equation

$$\frac{h^3}{12}\frac{d^4}{d\beta^4}\,D\varphi = \lambda\,(hD + \rho E)\,\varphi \tag{2.6}$$

later plays a governing role.

Let us consider th periodic problem for (2.6). Taking account of the second relation in (2.2), we see that (2.6) has a complete system of eigenfunctions $\exp(2\,n\pi i\beta/a)$ on Γ. Its double eigennumbers hence have the form

$$\lambda_n^{\pm} = (h^3/12)\,(2\,|\,n\,|\,\pi/a)^5\,[h\,(2\,|\,n\,|\,\pi/a) + \rho]^{-1} \tag{2.7}$$

It can be seen that the following assertion results from (2.7). There exist constants $\varepsilon > 0$ and $h_0 > 0$ such that for (2.8)

$$0 < h < h_0,\ \varepsilon < \{F(\lambda)\} < 1 - \varepsilon, \quad \left(F(\lambda) = \frac{a}{2\pi\mu}\left((\rho\lambda)^{1/5} + 12^{1/3}5^{-1}\mu^{2/3}\left(\frac{\lambda^2}{\rho^3}\right)^{1/5}\right),\quad \mu = \frac{h^{3/5}}{12^{1/5}}\right. \tag{2.8}$$

the eigennumber distribution function $n_0(\lambda)$ of the periodic problem for (2.6) has the form

$$n_0(\lambda) = 2\,[F(\lambda)] + 1 \tag{2.9}$$

3. Let $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ be eigennumber distribution functions of the problem (2.3) and the periodic problem for plane oscillations of a cylindrical shell with k waves along the generatrix.

Theorem 2. The ends of a fixed segment $I = [\lambda_1, \lambda_2]$ do not belong to the spectrum of the problems mentioned above. There exist constants $\varepsilon > 0$ and $h_0 > 0$ such that for $0 < h < h_0$ and $\varepsilon < \{F(\lambda_i)\} < 1 - \varepsilon$ the formula

$$n(\lambda_2) - n(\lambda_1) = 2\,F(\lambda_2) - 2\,F(\lambda_1) + \sigma_1(\lambda_2) - \sigma_1(\lambda_1) + \sigma_2(\lambda_2) - \sigma_2(\lambda_1) \tag{3.1}$$

is valid for the quantity of eigennumbers of the moment problem (2.5) on the segment I.

Proof. We assume that there are no eigennumbers of the problem (2.3) and the problem of plane oscillations with k waves along the generator in the segment I. Then (2.5) can be represented in the form

$$\frac{h^3}{12}\frac{d^4}{d\beta^4}\,D\varphi - \lambda\,(hD + \rho E)\,\varphi + h^3 A\varphi + hB\varphi = 0 \tag{3.2}$$

where the order of A is 3, the order of B is -1, and the operators A and B depend regularly on λ on the segment I. We consider the rectangle Π with sides parallel to the coordinate axes in a complex plane. Let the vertical sides pass through the points λ_1 and λ_2.

Let us substitute $\psi = (hD + \rho E)\,\varphi$ in (3.2). We obtain

$$P\,\psi - \lambda\psi + (h^3 A + hB)\,(hD + \rho E)^{-1}\psi = 0, \quad P = \frac{h^3}{12}\frac{d^4}{d\beta^4}\,D\,(hD + \rho E)^{-1} \tag{3.3}$$

Denoting the resolvent of the operator P by R_λ, we have an estimate for

$$\| R_\lambda B (hD + \rho E)^{-1} \| \leqslant c_1 / \mu, \ \| R_\lambda A (hD + \rho E)^{-1} \| \leqslant c_2 / \mu^2 \tag{3.4}$$

which can be obtained by taking into account the explicit form of the eigennumbers (2.7) of equation (2.6). Integrating the resolvent of (3.3) along the contour Π with (3.4) and the fact that the spectrum of the operator P agrees with the spectrum of (2.6) taken into account, we obtain (3.1).

Now, let there be one, for simplicity, a simple pole λ_* of the operator T_1 on the segment I, i.e., equation (2.5) has the following form for $\lambda \in I$

$$\frac{h^3}{12}\frac{d^4}{d\beta^4} D\varphi - \lambda (hD + \rho E)\varphi + (h^3 A + hB)\varphi + h\frac{f_1(s)(f_2, \varphi)}{\lambda_* - \lambda} = 0 \tag{3.5}$$

It can be shown that (3.5) is equivalent to the following integral equation with the meromorphic kernel

$$\psi + hf_1(s)(f_2, (hD + \rho E)^{-1} R_\lambda \psi) / (\lambda_* - \lambda) = 0 \tag{3.6}$$

The Fredholm determinant of (3.6) can be represented in the form

$$\lambda - \lambda_* + O(\mu^{2/3}) + \mu^{5/3} \sum_{\lambda_i(\mu) \in I} \frac{(f_1, y_i)(y_i, (hD + \rho E)^{-1} f_2)}{\lambda_i(\mu) - \lambda} = 0 \tag{3.7}$$

where $\lambda \in \Pi$, $\lambda_i(\mu)$ is the pole of R_λ on the segment I. Because of the smoothness of $f(s)$ the scalar product (f, y_i) decreases more rapidly than any power of μ as $\mu \to 0$. Hence, from the principle of the argument there results that in a sufficiently large rectangle the number of zeroes of (3.7) is one greater than the number of its poles $\lambda_i(\mu)$. Since the eigennumbers of (3.7) are real by virtue of Theorem 1, Theorem 2 is proved.

Remark 3°. The ends of the segment I decrease as $h \to 0$ and behave as h^α $(\alpha < 1)$. It can be verified that Theorem 2 remains valid even in this case, where only the first two components remain in the right side of (3.2). Let $\lambda = \lambda_1 h$. We represent (2.8) in the form

$$\frac{h^2}{12}\frac{d^4}{d\beta^4} D\varphi + K(0)\varphi + h[hA + T(\lambda_1) + K_1(\lambda_1)]\varphi = \lambda_1 \rho\varphi \tag{3.8}$$

If we set $h = 0$ in (3.8), a degenerate equation is obtained

$$K(0)\varphi = \lambda_1 \rho \varphi \tag{3.9}$$

that is equivalent to a "membrane" system of equations

$$\begin{gathered} -\left[\frac{1-\sigma}{2}\frac{d^2}{d\beta^2} - \varkappa^2\right] u - \frac{1+\sigma}{2}\varkappa\frac{dv}{d\beta} + \frac{1}{R(\beta)}\varkappa w = 0 \\ \frac{1+\sigma}{2}\varkappa u - \left[\frac{d^2}{d\beta^2} - \frac{1-\sigma}{2}\varkappa^2\right] v + \frac{d}{d\beta}\left(\frac{w}{R(\beta)}\right) = 0 \\ \frac{1}{R(\beta)}\varkappa u - \frac{1}{R(\beta)}\frac{dv}{d\beta} + \frac{1}{R^2(\beta)} w = \lambda_1 \rho T^{-1} w \end{gathered} \tag{3.10}$$

where T^{-1} is an integral operator inverse to the operator (2.2). The spectrum of the system of equations (3.10) is discrete and is concentrated at zero.

Let $\sigma_3(\lambda_1)$ denote the number of eigenvalues of the system (3.10) that are larger than λ_1. As before, let $\sigma_1(\lambda)$ be the distribution function of the eigenvalues of the Dirichlet problem of the Helmholtz equation in a domain bounded by the contour Γ, and $\sigma_2(\lambda)$ is the distribution function of the eigenvalues of the periodic problem for the equations of plane oscillations with waves along the generator.

We denote S as the combination of the spectra of these problems.

Theorem 3. There exist constants $\varepsilon > 0$ and $h_0 > 0$ such that for $0 < h \leqslant h_0$, $\varepsilon < F(\lambda) < 1 - \varepsilon$, $\lambda > \Lambda h$, $\operatorname{dist}(\lambda, S) > \varepsilon$, where $\Lambda > 0$ is any number, the distribution function of the natural frequencies $n(\lambda)$ of the problem (2.1) has the form

$$n(\lambda) = 2F(\lambda) + 1 + \sigma_1(\lambda) + \sigma_2(\lambda) - \sigma_3(\lambda / h) \tag{3.11}$$

Let R_{λ_1} denote the resolvent of the periodic problem on Γ for the equation

$$\frac{h^2}{12}\frac{d^4}{d\beta^4} D\varphi = \lambda_1 \rho\varphi \tag{3.12}$$

Then the resolvent $R_{\lambda_1}^{(1)}$ of the analogous problem for (3.8) can be represented in the form

$$R_{\lambda_1}^{(1)} = (E + hR_{\lambda_1}(hA + T(\lambda_1) + K_1(\lambda_1)) + R_{\lambda_1} K(0))^{-1} R_{\lambda_1} \tag{3.13}$$

It can be shown that

$$\| hR_{\lambda_1}(hA + T(\lambda_1) + K_1(\lambda_1)) \| \to 0, \quad \| R_{\lambda_1} K(0) + \lambda_1^{-1} K(0) \| \to 0$$
$$(h \to 0)$$

if λ_1 satisfies the conditions of Theorem 3. In the complex λ_1 plane we examine the closed contour Π passing through the point λ_1 on the positive real half-axis containing the segment $[0, \lambda_1]$ and satisfying the conditions of the theorem. Then on this contour

$$R_{\lambda_1}^{(1)} = \left(E - \frac{1}{\lambda_1} K(0) + o(h)\right)^{-1} R_{\lambda_1} \tag{3.14}$$

We denote $R_{\lambda_1}^{(0)}$ as the resolvent of (3.9). Using (3.14), we obtain that on the contour Π

$$\left\| R_{\lambda_1}^{(1)} - R_{\lambda_1} - R_{\lambda_1}^{(0)} - \frac{1}{\lambda_1} E \right\| \to 0 \quad (h \to 0) \tag{3.15}$$

Integrating (3.15) over the contour Π, we obtain (3.11) for the case $\lambda = \lambda_1 h$. The complete formula (3.11) is obtained taking Theorem 2 and Remark 3° into account.

In conclusion, we note that rapidly varying eigen vector-functions correspond to the first term in (3.11) in the gap between eigennumbers of the degenerate problem. Their components have the form

$$u = -\frac{\mu^2 \sigma}{F^2(\lambda) R(\beta)} \exp\left(iF(\lambda) \frac{\beta}{\mu}\right)(1 + O(\mu))$$
$$v = \frac{\mu}{iF(\lambda) R(\beta)} \exp\left(iF(\lambda) \cdot \frac{\beta}{\mu}\right)(1 + O(\mu))$$
$$w = \exp\left(iF(\lambda) \frac{\beta}{\mu}\right)(1 + O)(\mu)), \quad \varphi|_{\Gamma} = \frac{i\omega\mu}{|F(\lambda)|} \exp\left(iF(\lambda) \frac{\beta}{\mu}\right)(1 + O(\mu))$$
$$\varphi|_{\Omega} = \frac{i\omega\mu}{|F(\lambda)|} \exp\left(iF(\lambda) \frac{\beta}{\mu} - |F(\lambda)| \frac{\rho}{\mu}\right)(1 + O(\mu))\,\theta(\rho)$$

where ρ is the distance along the normal from the boundary of Γ and $\theta(\rho)$ is the cutoff function.

REFERENCES

1. ASLANIAN A.G., and LIDSKII V.B., Natural Frequency Distribution of Thin Elastic Shells. NAUKA, Moscow, 1974.

2. ODHNOFF J., Operators generated by differential problems with eigenvalue parameter in equation and boundary condition. Medd. Lunds Univ. Mat. Sem. Vol.14, 1959.

3. GOKHBERG I.Ts. and KREIN M.G., Introduction to the Theory of Linear Non-self-adjoint Operators in Hilbert Space. NAUKA, Moscow, 1965. (See also English translation , Theory and applications of Volterra operators in Hilbert space. Providence, American Mathematical Society, 1970).

4. HÖRMANDER L., Pseudo-differential operators, Commun. Pure and Appl. Math., Vol.18, No.3. 1965.

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